DANISH
TECHNOLOGICAL
INSTITUTE

A note on buoyancy correction

Danish Technological Institute

Title:
A note on buoyancy correction in the traceability chain for gravimetric flow measurement

Contact: Claus Melvad, tel.: +45 7220 2098, email: cmd@dti.dk

Prepared by:
Danish Technological Institute
Kongsvang Allé 29
8000 Aarhus C
Installation and Calibration

November 2013
Author: Claus Melvad, Morten Karstoft Rasmussen and John Frederiksen

This document is written with funding by the European Union under the EMRP. The EMRP is jointly funded by the EMRP participating countries within EURAMET and the European Union.

EMRP
European Metrology Research Programme
Programme of EURAMET

The EMRP is jointly funded by the EMRP participating countries within EURAMET and the European Union

Weighing equation including buoyancy

There tend to be some degree of confusion when discussing the terms of buoyancy, buoyant force, mass and weight. Consequently Figure 1 is included in order to illustrate the meaning and physics of these terms.

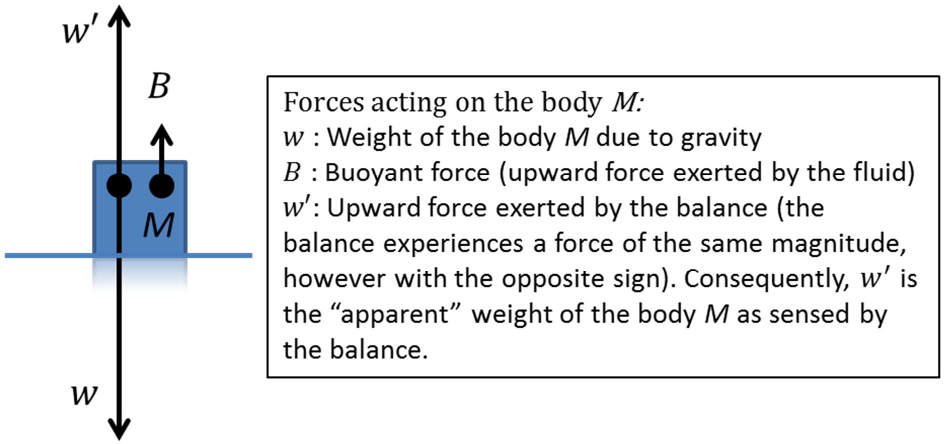

Figure 1: This figure illustrates a free-body diagram of a weight (body *M*) placed on a balance. All forces acting on the body *M* must sum up to zero since the weight stays at rest. Applying Newton's laws of motion we have: $w = w' + B$. The upward force exerted by the balance is described as: $w' = w - B = mg - mg\rho_a/\rho_M$, where m, ρ_M, ρ_a and g describe the mass of the body *M*, density of the body *M*, density of the surrounding air, and the acceleration due to gravity, respectively. Dividing with g yields: $w'/g = m(1 - \rho_a/\rho_M)$, which explains how the apparent weight w' of the body *M*, relates to the mass *m* of the body *M*. When balancing two bodies, you want to describe the equilibrium (balance) of the two bodies using a comparator which, as the figure illustrates, senses the "apparent" weight of the body *M*. Consequently, the right hand side of the previous equation is normally applied when describing a weighing process where a number of bodies are in equilibrium. In addition, this illustrates how the resulting weighing equation is independent of the magnitude if the local gravitational field.

The following equation describes a weighing process, where the balance's internal weight of mass m_{bal} and density ρ_{bal}, is in equilibrium with a reference weight of mass m_{wgt} and density ρ_{wgt}:

$$m_{\mathrm{bal}}\left(1 - \rho_{\mathrm{a_{cal}}}/\rho_{\mathrm{bal}}\right) = m_{\mathrm{wgt}}\left(1 - \rho_{\mathrm{a_{cal}}}/\rho_{\mathrm{wgt}}\right) \qquad (1)$$

The calibrated mass m_{bal} can then be calculated given that the reference mass m_{wgt} is known from e.g. a certificate:

$$m_{\mathrm{bal}} = m_{\mathrm{wgt}} \frac{\left(1 - \rho_{\mathrm{a_{cal}}}/\rho_{\mathrm{wgt}}\right)}{\left(1 - \rho_{\mathrm{a_{cal}}}/\rho_{\mathrm{bal}}\right)} \tag{2}$$

If the weight densities are equal, that is $\rho_{\mathrm{wgt}} = \rho_{\mathrm{bal}}$, which is often the case, then equation (2) tells us that the two masses can be compared directly if the weighing of the two masses are performed close together in time.

Equations for dispensed water measurement corrected for buoyancy

Using this balance to measure a mass of an object is performed by comparing the balance's internal mass with the mass of the object at that specific air density.

First considering the buoyancy of the beaker, water, oil, and balance weight before measurement, where O_1 is the offset of the balance reading with an empty pan:

$$m_{\mathrm{bal_1}}\left(1 - \rho_{\mathrm{a_1}}/\rho_{\mathrm{bal_1}}\right) = m_{\mathrm{w_{pre}}}\left(1 - \rho_{\mathrm{a_1}}/\rho_{\mathrm{w_1}}\right) + m_{\mathrm{oil}}\left(1 - \rho_{\mathrm{a_1}}/\rho_{\mathrm{oil_1}}\right) + m_{\mathrm{beaker}}\left(1 - \rho_{\mathrm{a_1}}/\rho_{\mathrm{beaker_1}}\right) + O_1 \tag{3}$$

Solving for $m_{w_{pre}}$ (mass of pre filled water already in the beaker) and introducing a number of constants for convenience:

$$m_{\mathrm{w_{pre}}} = m_{bal_1} \frac{K_{\mathrm{bal_1}}}{K_{\mathrm{w_1}}} - m_{oil} \frac{K_{\mathrm{oil_1}}}{K_{\mathrm{w_1}}} - m_{\mathrm{beaker}} \frac{K_{\mathrm{beaker_1}}}{K_{\mathrm{w_1}}} - \frac{O_1}{K_{\mathrm{w_1}}} \tag{4}$$

Where $K_{\mathrm{bal_1}} = \left(1 - \rho_{\mathrm{a_1}}/\rho_{\mathrm{bal_1}}\right)$, $K_{\mathrm{w_1}} = \left(1 - \rho_{\mathrm{a_1}}/\rho_{\mathrm{w_1}}\right)$, $K_{\mathrm{oil_1}} = \left(1 - \rho_{\mathrm{a_1}}/\rho_{\mathrm{oil_1}}\right)$ and $K_{\mathrm{beak_1}} = \left(1 - \rho_{\mathrm{a_1}}/\rho_{\mathrm{beaker_1}}\right)$.

Then a certain mass of water, m_w, is dispensed and meanwhile also the air density changes typically due to a change in the air pressure. A change in temperature might also change the density of the balance beaker, water, oil and weight. Furthermore, the zero point offset may have changed. This leads to a new equilibrium described by the following weighing equation:

$$m_{\mathrm{bal_2}}\left(1 - \rho_{\mathrm{a_2}}/\rho_{\mathrm{bal_2}}\right) = m_{\mathrm{w_{pre}}}\left(1 - \rho_{\mathrm{a_2}}/\rho_{\mathrm{w_2}}\right) + m_{\mathrm{oil}}\left(1 - \rho_{\mathrm{a_2}}/\rho_{\mathrm{oil_2}}\right) + m_{\mathrm{w}}\left(1 - \rho_{\mathrm{a_2}}/\rho_{\mathrm{w_2}}\right) + m_{\mathrm{beaker}}\left(1 - \rho_{\mathrm{a_2}}/\rho_{\mathrm{beaker_2}}\right) + O_2 \tag{5}$$

Equation (5) assumes that the temperature of the dispensed water and the water already present in the beaker are equal, which is reasonable given the stable operating environment. Inserting $m_{w_{pre}}$ from equation (4) into equation (5), introducing again a number of constants and solving for m_{w} yields:

$$m_{\mathrm{w}} = m_{\mathrm{bal}_2}\frac{K_{\mathrm{bal}_2}}{K_{\mathrm{w}_2}} - m_{\mathrm{bal}_1}\frac{K_{\mathrm{bal}_1}}{K_{\mathrm{w}_1}} + m_{\mathrm{oil}}\left(\frac{K_{\mathrm{oil}_1}}{K_{\mathrm{w}_1}} - \frac{K_{\mathrm{oil}_2}}{K_{\mathrm{w}_2}}\right) + m_{\mathrm{beaker}}\left(\frac{K_{\mathrm{beaker}_1}}{K_{\mathrm{w}_1}} - \frac{K_{\mathrm{beaker}_2}}{K_{\mathrm{w}_2}}\right) + \frac{O_1}{K_{\mathrm{w}_1}} - \frac{O_2}{K_{\mathrm{w}_2}} \tag{6}$$

Where $K_{\mathrm{bal}_2} = \left(1 - \rho_{\mathrm{a}_2}/\rho_{\mathrm{bal}_2}\right)$, $K_{\mathrm{w}_2} = \left(1 - \rho_{\mathrm{a}_2}/\rho_{\mathrm{w}_2}\right)$, $K_{\mathrm{oil}_2} = \left(1 - \rho_{\mathrm{a}_2}/\rho_{\mathrm{oil}_2}\right)$ and $K_{\mathrm{beaker}_2} = \left(1 - \rho_{\mathrm{a}_2}/\rho_{\mathrm{beaker}_2}\right)$.

Examples of assumptions to simplify equation (6):

- Assuming the offset error of the balance is the same before and after ($O_1 = O_2$).
- Assuming zero offset error of the balance ($O_1 = O_2 = 0$).
- Assuming zero change in water density with a water temperature change ($\rho_{\mathrm{w}_1} = \rho_{\mathrm{w}_2}$).
- Assuming zero change in balance weight density ($\rho_{\mathrm{bal}_1} = \rho_{\mathrm{bal}_2}$).
- Assuming zero change in oil density ($\rho_{\mathrm{oil}_1} = \rho_{\mathrm{oil}_2}$).
- Assuming zero change in glass density ($\rho_{\mathrm{beaker}_1} = \rho_{\mathrm{beaker}_2}$).
- Assuming the mass of oil to be zero ($m_{\mathrm{oil}} = 0$).

These assumptions lead to a negligible error for the setup at DTI, however, this may not be the case in general as described by equation (6). The above assumptions lead to a slightly simpler equation for the dispensed water, m_w, when taking into account the exact buoyancy on objects with different densities:

$$m_{\mathrm{w}} = m_{\mathrm{bal}_2}\frac{K_{\mathrm{bal}_2}}{K_{\mathrm{w}_2}} - m_{\mathrm{bal}_1}\frac{K_{\mathrm{bal}_1}}{K_{\mathrm{w}_1}} + m_{\mathrm{beaker}}\left(\frac{K_{\mathrm{beaker}_1}}{K_{\mathrm{w}_1}} - \frac{K_{\mathrm{beaker}_2}}{K_{\mathrm{w}_2}}\right) \tag{7}$$

Equations for dispensed water measurement corrected for buoyancy (based on conventional mass inputs)

Occasionally, the conventional masses are given instead of true masses, and in this case a similar equation describing the mass of the dispensed water can be derived. Setting up weighing equations before and after dispensing water, yields two equations analogous to equation (3) and (5), however now expressed by applying the conventional masses instead. In order not to confuse the concepts of true mass and conventional mass, the symbol cm is used here for conventional mass. Consequently, the conventional mass of the prefilled water already in the beaker, cm_{w_1}, is described by the following equilibrium weighing equation (the balance offset is not accounted for here, see the list off assumptions in the previous section which are also applied here):

$$cm_{\mathrm{w}_1} = cm_{\mathrm{bal}_1}\left(1 + C_{\mathrm{bal}_1}\right) - cm_{\mathrm{oil}_1}\left(1 + C_{\mathrm{oil}_1}\right) - cm_{\mathrm{beaker}_1}\left(1 + C_{\mathrm{beaker}_1}\right) \quad (8)$$

Where $C_{\mathrm{bal}_1} = \frac{(\rho_{\mathrm{bal}_1} - \rho_{\mathrm{w}_1})(\rho_{\mathrm{a}_1} - \rho_{\mathrm{a}_0})}{(\rho_{\mathrm{bal}_1} - \rho_{\mathrm{a}_0})(\rho_{\mathrm{w}_1} - \rho_{\mathrm{a}_1})}$, $C_{\mathrm{oil}_1} = \frac{(\rho_{\mathrm{oil}_1} - \rho_{\mathrm{w}_1})(\rho_{\mathrm{a}_1} - \rho_{\mathrm{a}_0})}{(\rho_{\mathrm{oil}_1} - \rho_{\mathrm{a}_0})(\rho_{\mathrm{w}_1} - \rho_{\mathrm{a}_1})}$, $C_{\mathrm{beaker}_1} = \frac{(\rho_{\mathrm{beaker}_1} - \rho_{\mathrm{w}_1})(\rho_{\mathrm{a}_1} - \rho_{\mathrm{a}_0})}{(\rho_{\mathrm{beaker}_1} - \rho_{\mathrm{a}_0})(\rho_{\mathrm{w}_1} - \rho_{\mathrm{a}_1})}$ and $\rho_{\mathrm{a}_0} = 1.2\ \mathrm{kg/m^3}$

Likewise, the following equation describes the weighing equilibrium after dispensing water, with cm_{w_2} denoting the total conventional mass of water in the beaker:

$$cm_{\mathrm{w}_2} = cm_{\mathrm{bal}_2}\left(1 + C_{\mathrm{bal}_2}\right) - cm_{\mathrm{oil}_2}\left(1 + C_{\mathrm{oil}_2}\right) - cm_{\mathrm{beaker}_2}\left(1 + C_{\mathrm{beaker}_2}\right) \quad (9)$$

Where $C_{\mathrm{bal}_2} = \frac{(\rho_{\mathrm{bal}_2} - \rho_{\mathrm{w}_2})(\rho_{\mathrm{a}_2} - \rho_{\mathrm{a}_0})}{(\rho_{\mathrm{bal}_2} - \rho_{\mathrm{a}_0})(\rho_{\mathrm{w}_2} - \rho_{\mathrm{a}_2})}$, $C_{\mathrm{oil}_2} = \frac{(\rho_{\mathrm{oil}_2} - \rho_{\mathrm{w}_2})(\rho_{\mathrm{a}_2} - \rho_{\mathrm{a}_0})}{(\rho_{\mathrm{oil}_2} - \rho_{\mathrm{a}_0})(\rho_{\mathrm{w}_2} - \rho_{\mathrm{a}_2})}$ and $C_{\mathrm{beaker}_2} = \frac{(\rho_{\mathrm{beaker}_2} - \rho_{\mathrm{w}_2})(\rho_{\mathrm{a}_2} - \rho_{\mathrm{a}_0})}{(\rho_{\mathrm{beaker}_2} - \rho_{\mathrm{a}_0})(\rho_{\mathrm{w}_2} - \rho_{\mathrm{a}_2})}$

Now the conventional mass of the dispensed water can be calculated by subtracting equation (8) from equation (9), and by assuming that $cm_{\mathrm{oil}_1} = cm_{\mathrm{oil}_2} = cm_{\mathrm{oil}}$ and $cm_{\mathrm{beaker}_1} = cm_{\mathrm{beaker}_2} = cm_{\mathrm{beaker}}$ (indicating that the oil and beaker densities before and after water dispensing are equal):

$$cm_{\mathrm{w}} = cm_{\mathrm{w}_2} - cm_{\mathrm{w}_1} = cm_{\mathrm{bal}_2}\left(1 + C_{\mathrm{bal}_2}\right) - cm_{\mathrm{bal}_1}\left(1 + C_{\mathrm{bal}_1}\right) + cm_{\mathrm{oil}}\left(C_{\mathrm{oil}_1} - C_{\mathrm{oil}_2}\right) + cm_{\mathrm{beaker}}\left(C_{\mathrm{beaker}_1} - C_{\mathrm{beaker}_2}\right) \quad (10)$$

Please notice that equation (10) only applies if the oil, beaker and water ($\rho_{w_1} = \rho_{w_2} = \rho_w$) densities can be assumed to be the same before and after dispensing water. If these assumptions are not valid, then a more general solution must be obtained from equation (8) and (9). The above equation (10) describes the conventional mass of the dispensed water, however, only the true mass is useful when calculating the flow. The mass of dispensed water can then be calculated from the conventional mass by applying the following equation:

$$m_{\mathrm{w}} = cm_{\mathrm{w}} \left(\frac{1 - \rho_{\mathrm{a}_0}/\rho_0}{1 - \rho_{\mathrm{a}_0}/\rho_{\mathrm{w}}} \right) \tag{11}$$

Where $\rho_{\mathrm{a}_0} = 1.2\ \mathrm{kg/m^3}$ and $\rho_0 = 8000\ \mathrm{kg/m^3}$.

References:

Frank E. Jones, Randall M. Schoonover, “Handbook of Mass Measurement”, CRC Press, 2002

Randall M. Schoonover, Frank E. Jones, “Air Buoyancy Correction in High-Accuracy Weighing on Analytical Balances”, Anal. Chem. 1981, 53, 900-902

OIML D 28, Edition 2004 E, “Conventional value of the result of weighing in air”